References

[1] Liepmann, H. W. and Roshko, A., *Elements of Gasdynamics* (John Wiley and Sons, Inc., New York, 1957), p. 172.

[2] Kovasznay, L. S. G., "Physical measurements in gas dynamics," *High Speed Aerodynamics and Jet Propulsion* (Princeton University Press, Princeton, N. J., 1954), Vol. IX, Sec. F-2.

[3] Kovasznay, L. S. G., "Hot-wire investigation of the wake behind cylinders at low Reynolds numbers," Proc. Roy. Soc. (London) **A198,** 174 (1948).

[4] Sato, H., "Non-linearity and inversion effects of hot-wire anemometer on the mean- and fluctuating-velocity measurements at low wind-speed," Rept. Inst. Sci. Technol. Univ. Tokoyo **11,** 73 (1957).

Minimum Length MHD Accelerator with Constant Enthalpy

HAROLD MIRELS,* RICHARD R. GOLD,† AND JAMES F. MULLEN‡
Aerospace Corporation, El Segundo, Calif.

In a previous note,[1] an analytical solution was presented for the minimum length of a crossed-field magnetohydrodynamic (MHD) accelerator with specified inlet and exit conditions. It was assumed that the flow was one-dimensional, that the working fluid was a perfect gas, that the enthalpy, magnetic field B, and electrical conductivity σ were each constant, and that the local joule heating was small as compared with the net local electrical power input ($\epsilon R \ll 1$). This solution has been generalized to include variable magnetic field and electrical conductivity. In particular, it has been assumed that

$$\sigma B^2 = \rho^{1-N} \tag{1}$$

where ρ is density and N is a constant. This generalization is outlined here, using the same notation as in Refs. 1 and 2. A brief comparison, with a related analytical solution in Ref. 3, is also made.

Substitution of Eq. (1) into Eq. (7) of Ref. 1 gives

$$x_2 = -\int_1^{u_2} \rho^N \frac{\rho u}{p_1\rho'}\left[1 + \frac{p_1\rho'}{\rho u}\right]^2 du \tag{2}$$

which is to be minimized. Equation (2) can be reduced to the same equation that is minimized in Ref. 1 if ρ^N and p_1/N are replaced by ρ and p_1, respectively. Hence, Eqs. (8–14) in Ref. 1 are directly applicable to the present problem provided that ρ and p_1 therein are replaced by ρ^N and p_1/N. If terms of order ϵR are neglected,§ then the density variation for a minimum-length accelerator is

$$\rho^N = [1 + \tfrac{2}{3}(N\epsilon/p_1)(u^{3/2} - 1)]^{-1} \tag{3}$$

and the corresponding variation of accelerator length with u is

$$x = (2/3\epsilon)(u^{3/2} - 1) \tag{4}$$

The other dependent variables, to order $\epsilon R \ll 1$, are

$$\left.\begin{array}{ll} E_y = uB(1 + \epsilon\rho^N/u^{1/2}) & E_x = Bj/\sigma\rho = \epsilon(u)^{1/2}/\sigma \\ j = \rho u^{1/2}\epsilon/B & \omega_e\tau_e = B/\rho \\ {}_jE_yA = \epsilon u^{1/2} & \Phi \equiv -\int_0^x E_x dx = -\int_1^u \frac{u}{\sigma} du \end{array}\right\} \tag{5}$$

The area variation is

$$A = 1/\rho u = [1 + (N\epsilon/p_1)\epsilon x]^{1/N}/(1 + \tfrac{3}{2}\epsilon x)^{2/3} \tag{6}$$

For small and large values of ϵx, respectively,

$$A = 1 + [(\epsilon/p_1) - 1]\epsilon x + O(\epsilon x)^2 \tag{7a}$$

$$= [(N\epsilon/p_1)^{1/N}/(\tfrac{3}{2})^{2/3}](\epsilon x)^{(3-2N)/3N}[1 + O(\epsilon x)^{-1}] \tag{7b}$$

A practical accelerator generally will require $dA/dx \geq 0$. This will require $\epsilon/p_1 \geq 1$ for small ϵx and $N \leq \frac{3}{2}$ for large ϵx.

If the accelerator exit conditions are specified, the required value of ϵ/p_1 is [from Eq. (3)]

$$\epsilon/p_1 = (3/2N)[(\rho_2^{-N} - 1)/(u_2^{3/2} - 1)] \tag{8}$$

The minimum length to achieve these conditions is

$$x_2 = \tfrac{4}{9}(N/p_1)[(u_2^{3/2} - 1)^2/(\rho_2^{-N} - 1)] \tag{9}$$

Equation (9) is of major interest, since it describes the minimum length in terms of the inlet and exit conditions, as well as N. The length decreases as N increases.

The variation of conductivity with density for a gas at constant temperature can be approximated generally by $\sigma = \rho^{-N_1}$. For a slightly ionized gas, σ is essentially proportional to the degree of ionization. The Saha equation then indicates $0 \leq N_1 \leq \frac{1}{2}$. For a more highly ionized gas, σ increases slightly with ρ so that N_1 becomes negative. A typical plot of σ vs ρ is shown in Fig. 1 for air seeded with potassium. These results were obtained by G. L. Johnston of Aerospace Corporation and permit values of N_1 to be obtained for various temperature and density regimes.

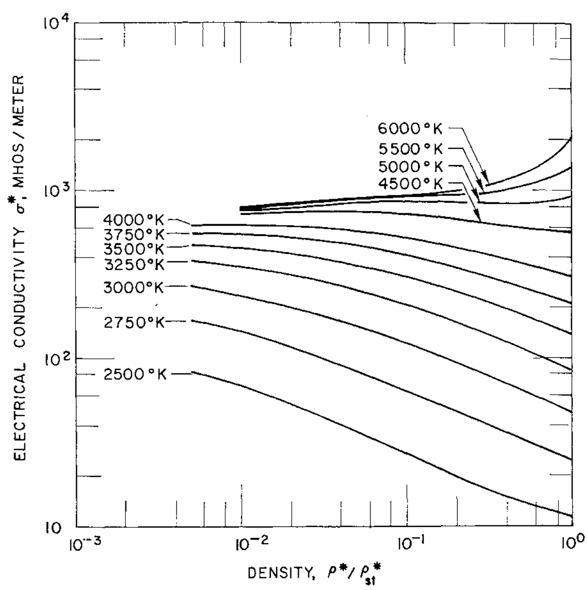

Fig. 1 Electrical conductivity of air seeded with 0.75% by weight of potassium (symbol ρ_{st}* denotes air density at STP).

Received December 4, 1964; revision received February 18, 1965. This research was supported by the U. S. Air Force under Contract No. AF04(695)-269.

* Head, Advanced Propulsion and Fluid Mechanics Department, Aerodynamics and Propulsion Laboratory. Associate Fellow Member AIAA.

† Head, Magnetogasdynamics Section, Aerophysics Department, Aerodynamics and Propulsion Laboratory; now at Hughes Aircraft Co., Space Systems Division. Associate Fellow Member AIAA.

‡ Member, Technical Staff, Advanced Propulsion and Fluid Mechanics Department, Aerodynamics and Propulsion Laboratory. Member AIAA.

§ A closed-form solution for minimum length can be obtained, without assuming $\epsilon R \ll 1$, for the special case $N = 0$. The resulting solution is

$$\epsilon R = (1 + Cu)^{-1/2}$$

$$\ln\rho = (2/3p_1C^2)[(2 - Cu)(1 + Cu)^{1/2} - (2 - C)(1 + C)^{1/2}]$$

$$x = (2/3C)[(1 + Cu)^{3/2} - 3Cu + 3(1 + Cu)^{1/2}]_{u=1}^{u=u}$$

where $C = (1 - \epsilon^2)/\epsilon^2$.

Assuming $\sigma = \rho^{-N_1}$, the present solution then applies for a magnetic-field variation of the form $B = \rho^{(1+N_1-N)/2}$. A constant B field solution corresponds to $N = 1 + N_1$, whereas a constant $\omega_e\tau_e$ solution corresponds to $B = \rho$ or $N = N_1 - 1$. To achieve the shortest accelerator length, consistent with an upper limit on $\omega_e\tau_e$, the accelerator design may require that B be uniform and as large as possible in the upstream portion of the accelerator and that $\omega_e\tau_e$ be constant at the maximum allowable value in the downstream portion. Such an accelerator can be designed by the piecewise application of the present analytical results.

Ring[3] also has presented a closed-form solution for a minimum-length constant enthalpy accelerator subject to the assumption $\sigma B^2 = \rho^{1-N}$. His solution is different from the one derived herein principally because of his use of an assumption other than $\epsilon R \ll 1$. Ring's results are summarized so that a meaningful comparison of the two solutions can be made. Ring obtained an expression for minimum length which is valid for all ϵR. This result is¶

$$x = \frac{1-\epsilon^2}{\epsilon^2}\frac{p_1}{N}\left[\frac{\rho^N}{u}\frac{1-\epsilon^2 R}{R^2(1-\epsilon^2)} - 1\right] \tag{10}$$

It is necessary to know ρ and R as functions of u in order to find x_2 as a function of inlet and exit conditions. To determine $R = R(u)$, Ring wrote Eq. (8) of Ref. 1 in the form

$$dR/du = -(1/K)(N\epsilon/p_1)uR^2(1-\epsilon R) \tag{11}$$

where

$$1/K = 1 + (p_1/2N\epsilon)[(1+\epsilon R)/Ru^2] \tag{12}$$

and assumed that a mean value of K can be used in Eq. (11). Successive integrations, without any limit on the magnitude of ϵR, gave

$$\frac{1}{R} - \epsilon \ln\left[\frac{(1-\epsilon)R}{1-\epsilon R}\right] = 1 + \frac{1}{K}\frac{N\epsilon}{p_1}\frac{u^2-1}{2} \tag{13a}$$

$$\rho = \{[(1-\epsilon)R]/(1-\epsilon R)\}^{K/N} \tag{13b}$$

Equations (10) and (13) define Ring's solution. Note that $1/K = 1 + (p_1/2N\epsilon)(1+\epsilon)$ at $u = 1$, whereas $1/K \rightarrow \frac{4}{3}$ as $u \rightarrow \infty$.

The form of the present solution is different from that of Ring's, except in the limit $u \rightarrow \infty$. The present solution represents a consistent expansion, which is valid everywhere to order $\epsilon R \ll 1$. Rin· does not require that $\epsilon R \ll 1$, but he does require the use of an appropriate mean value for K. It would appear that the present solution is preferable when $\epsilon R \ll 1$, whereas Ring's solution is useful where ϵR is not small. Since ϵR represents the ratio of joule heating to net local energy input, most practical accelerator designs will require that ϵR be small.

The area variation for the present class of accelerators may be difficult to fabricate and may not permit shock-free supersonic flow. Hence, these solutions should be considered as providing a first estimate for a physically realistic minimum-length accelerator.

References

[1] Mirels, H., Gold, R. R., and Mullen, J. F., "Minimum length MHD accelerator with constant enthalpy and magnetic field," AIAA J. 2, 1141–1143 (1964).

[2] Mirels, H., "Analytical solution for constant enthalpy MHD accelerator," AIAA J. 2, 145–146 (1964).

[3] Ring, L. E., "Optimization of MHD crossed-field accelerators and generators," Fifth Symposium on Engineering Aspects of Magnetohdrodynamics, Massachusetts Institute of Technology, Cambridge, Mass. (April 1–2, 1964).

¶ This can be found most easily by letting $\rho \equiv e^{-s}$ and $\rho'/\rho = -s'$, substituting into the integrand of Eq. (2), and optimizing with respect to s.

More General Solutions of the Incompressible Boundary-Layer Equations ($Pr = 1$)

William C. Meecham*
University of Minnesota, Minneapolis, Minn.

Nomenclature

C_p = specific heat at constant pressure
D/Dt = $u\partial/\partial x + v\partial/\partial y$
f = Blasius function
k = thermal conductivity
μ = fluid viscosity
ν = kinematic viscosity, μ/ρ
p = freestream pressure
R = universal gas constant
ρ = fluid density
T = fluid temperature
T_1 = freestream temperature
u = tangential velocity
u_y = $\partial u/\partial y$, etc.
U_1 = freestream velocity
v = transverse velocity
x = tangential coordinate
y = transverse coordinate

Introduction

Crocco[1] first produced the now familiar result that the temperature of a fluid is a quadratic function of u in two-dimensional, laminar, steady, compressible, thermal boundary-layer flow when $Pr = 1$. It is the purpose of this note to show that, if one restricts the problem to incompressible flow with constant k and μ, then Cu_y can be added to the expression for the temperature, where C is an arbitrary constant. The equations for the described compressible boundary-layer flow are[2]

$$(\partial/\partial x)(\rho u) + (\partial/\partial y)(\rho v) = 0 \tag{1}$$

$$\rho(D/Dt)u = (\partial/\partial y)(\mu u_y) - p_x \tag{2}$$

$$\rho C_p(D/Dt)T = (\partial/\partial y)(kT_y) + \mu u_y^2 + up_x \tag{3}$$

$$\frac{p}{\rho} = RT \tag{4}$$

When $Pr = 1$, a solution of these equations for T is

$$T = -(1/2C_p)u^2 + C_1 u + C_2 \tag{5}$$

where C_1 and C_2 are constants, and $C_1 = 0$ if $p_x \neq 0$.

Analysis and Discussion

If one further specializes the problem to incompressible flow with μ and k constant, it is easily shown, using Eq. (1), that

$$\rho(D/Dt)u_y = \mu(u_y)_{yy} \tag{6}$$

It is found, using Eq. (6), in this case, that

$$T = -(1/2C_p)u^2 + C_1 u + C_2 + Cu_y \tag{7}$$

is a solution of Eq. (3), where again $C_1 = 0$ if $p_x \neq 0$. To find u (and u_y) as in the use of Eq. (5), one must solve the velocity equations [Eqs. (1) and (2)].

Consider the boundary-layer flow over a curved surface. Then $p_x \neq 0$, and $C_1 = 0$ in Eq. (7). The temperature at the

Received March 18, 1965. It is a pleasure to acknowledge very helpful discussions with W. H. Warner and James Serrin in connection with this work.

* Professor, Department of Aeronautics and Engineering Mechanics.